Linear Algebra II

11/04/2011, Monday, 13:00-16:00

1 **Gram-Schmidt process**

Let V be a real inner product space. Also let $\{u, v, w\} \subset V$ be a set of linearly independent vectors such that

$$\|u\| = \|v\| = \|w\| = 1$$

and

$$\langle u, v\rangle = \frac{3}{5} \quad \langle v, w\rangle = \frac{12}{25} \quad \langle w, u\rangle = \frac{4}{5}.$$

(a) Is the set $\{u, v, w\}$ an orthogonal set?

(b) Find $\|5u - 3v\|$.

(c) Find an orthonormal basis for the subspace $\text{span}(\{u, v, w\})$.

2 **Diagonalization**

Consider the matrix

$$M = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}.$$

For which values of (a, b, c) is this matrix

(a) diagonalizable?

(b) diagonalizable by a unitary matrix?

(c) diagonalizable by an orthogonal matrix?

3 **Singular value decomposition**

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix.

(a) Show that A is normal.

(b) Show that all eigenvalues of A are real.

(c) Show that $|\lambda|$ is a singular value of A if λ is an eigenvalue of A.

(d) Find a singular value decomposition of A in terms of its eigenvalues and orthogonal diagonalizer.

(e) Find the best rank k approximation of A.

4 Positive definiteness

(a) Consider the function
$$f(x, y) = \sin(x) + y^3 + 3xy + 2x - 3y.$$

(i) Show that $(0, -1)$ is a stationary point.

(ii) Determine whether this point corresponds to a local minimum, maximum, or saddle point.

(b) Let A be a symmetric matrix. Show that e^A is symmetric and positive definite.

5 Cayley-Hamilton theorem

Let $A \in \mathbb{R}^{2\times 2}$ be a matrix with the characteristic matrix $p_A(\lambda) = \lambda^2 - \lambda - 1$. Let
$$\alpha_0 = \alpha_1 = 1$$
and
$$\alpha_{k+2} = \alpha_{k+1} + \alpha_k$$
for $k \geq 0$. Show that
$$A^{n+2} = \alpha_{n+1}A + \alpha_n I$$
for $n \geq 0$. [Hint: Induction!]

6 Jordan canonical form

Find the Jordan canonical form J and determine a matrix X such that $X^{-1}AX = J$ for the matrix
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Each problem is 15 points: 6×15 + 10 (gratis)=100.